Cohomology of Arithmetic Groups

① General & Special Linear Groups $Gl_n \mathbb{Z}$, $Sl_n \mathbb{Z}$

② The Symplectic Group $Sp_{2n} \mathbb{Z}$

Subgroup of $Sl_{2n}\mathbb{Z}$ that preserves "standard symplectic form"

$$\mathbb{Z}^{2n} = \langle e_1, e_2, .., e_n, \bar{e}_1, .., \bar{e}_n \rangle$$

$$\omega(e_i, \bar{e}_j) = \delta_{ij} = -\omega(\bar{e}_j, e_i)$$

$$\omega(e_i, e_j) = 0 = \omega(\bar{e}_i, \bar{e}_j)$$

$$\begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix}$$

n : "genus"

Note : $Sp_{2,1} = SL_2$

③ Congruence Subgroups

$$\Gamma_n(p) : \ker\left(Sp_{2n}\mathbb{Z} \xrightarrow{\text{mod } p} Sp_{2n}\mathbb{F}_p\right)$$

Group Cohomology : $H^*(G; \mathbb{Q})$: H^* of a "classifying space" for G.

For $G <_{f.i.} Sp_{2n}\mathbb{Z}$, have a top "vcd" $= n^2$ beyond which $H^*(G;\mathbb{Q}) = 0$.

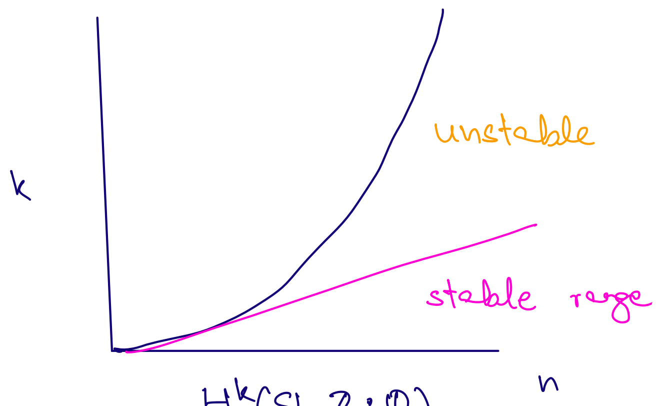
($\rightsquigarrow$ model for classifying space of dim n^2)

For $G <_{f.i.} Gl_n\mathbb{Z}$, vcd $= \binom{n}{2}$

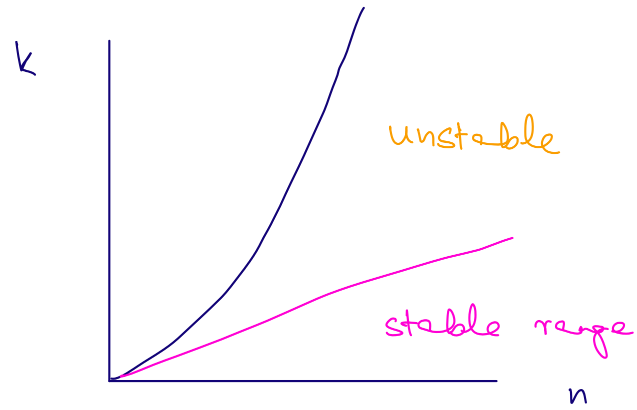
Note: For this talk, fix the $\mathbb{Z}$ in $Sp_{2n}\mathbb{Z}$.

But everything can be done in more generality.

A tool for computations : The Steinberg Module



$H^k(SL_n\mathbb{Z};\mathbb{Q})$



$H^k(Sp_{2n}\mathbb{Z};\mathbb{Q})$

Unstable range : Difficult!
Know there is lots of cohomology, but don't know many actual classes

Thm [Borel-Serre, Bieri-Eckmann]

for $G <_{f.i} SL_n\mathbb{Z}$, we have $H^{\binom{n}{2}-i}(G;\mathbb{Q}) \cong H_i(G;\underbrace{St_n\mathbb{Q}}_{\text{"Steinberg Module"}})$

For $G <_{f.i.} Sp_{2n}\mathbb{Z}$, we have

$$H^{n^2-i}(G;\mathbb{Q}) \cong H_i(G;\underbrace{St^{\omega}_{2n}\mathbb{Q}}_{\text{"Symplectic Steinberg Module"}})$$

Will define Steinberg modules later...

For now:

- It is the top homology of the Tits building ($\simeq$ combinatorial model for ∂ of sym. spaces for SL, Sp)
- Can define $St_n\mathbb{F}$, $St^{\omega}_{2n}\mathbb{F}$ for any field. For finite $\mathbb{F}$, we know the rank.

- Compute $H_i(G; St^{\omega}_{2n}\mathbb{Q})$ by building free/flat G-resolutions of $St^{\omega}_{2n}\mathbb{Q}$, then take G-coinvariants.

$$\cdots \to F_2 \to F_1 \to F_0 \to St^{\omega}_{2n} \to 0$$

$$\cdots \to (F_2)_G \to (F_1)_G \to (F_0)_G \to 0$$

- Often these (partial) resolutions are built by constructing suitable simplicial complexes and proving high-connectivity results on them.

(Why high connectivity? Eg: • Vanishing H_* groups $\Rightarrow$ exactness of resolution

• Know some page of a s.s. and know it converges to 0, that can give exactness etc.)

- Larger resolutions : easier to construct, but harder to use

Smaller resolutions : (very) hard to construct, easier to use

Depending on intended applications, both can have their uses.

<u>Thm [Lee - Szczarba, '76]</u> Construction of flat resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to St_n\mathbb{Q} \to 0$$

$$F_k = \bigoplus [\underbrace{L_1, L_2, \ldots, L_{n+k}}_{\text{spanning set of lines}}] \qquad \partial : \text{alternating sum}$$

<u>Thm [Lee-Szczarba]</u> $H^{\binom{n}{2}}(\Gamma_n^{SL}(3); \mathbb{Q}) \cong H_0(\Gamma_n(3); St_n\mathbb{Q})$

$$\cong (St_n\mathbb{Q})_{\Gamma_n(3)} \cong St_n(\mathbb{F}_3)$$

$$\text{rank } St_n\mathbb{F}_3 = 3^{\binom{n}{2}}$$

- In 2024, Ash-Miller-Patzt established a Hopf algebra structure on $\bigoplus_{k,n} H_k(GL_n \mathbb{Z}; St_n)$ using this resolution.

Independently, [Brown-Chan-Galatius-Payne, 2024] also proved a Hopf algebra structure, using very different methods, used this to conclude a lot more about $H^*(GL_n \mathbb{Z}; \mathbb{Q})$.
(eg, growth of the sizes of H^*)

Rmk: Both Hopf alg. structures are believed to be equivalent, but this has not been proven yet.

- Lee-Szczarba's construction fails for $St^\omega_{2n}\mathbb{Q}$.

Motivation for our resolution: • Hoping to investigate algebraic structures on $\bigoplus_{k,n} H_k(Sp_{2n}\mathbb{Z}; St^\omega_{2n})$

Thm (P.) Construction of a flat resolution of $St^\omega_{2n}\mathbb{F}$ for field $\mathbb{F}$.

Thm (P.) $H^{n^2}(\Gamma^{sp}_n(3); \mathbb{Q}) \cong (St^\omega_{2n}\mathbb{Q})_{\Gamma_n(3)}$
$\cong St^\omega_{2n}(\mathbb{F}_3)$] Top H_* of Tits building $T^\omega_{2n}\mathbb{F}_3$

rank $St^\omega_{2n}\mathbb{F}_3 = 3^{n^2}$

Proved this by specialising the resolution of $St^\omega_{2n}\mathbb{F}_3$ to a presentation.

This proof & theorem are false for prime levels $p \geq 5$ by, for eg, work of [Caporilla-Searle].

Crucial Piece: Only units in $\mathbb{F}_3$ are ± 1, which are also units in $\mathbb{Z}$.

(Symplectic) Steinberg Module

$\mathbb{T}^{\omega}_{2n}\mathbb{Q}$: Symplectic Tits Building

$(\mathbb{Q}^{2n}, \omega)$ ω: Symplectic form

$\quad \shortparallel$

$\langle e_1, e_2, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n \rangle$

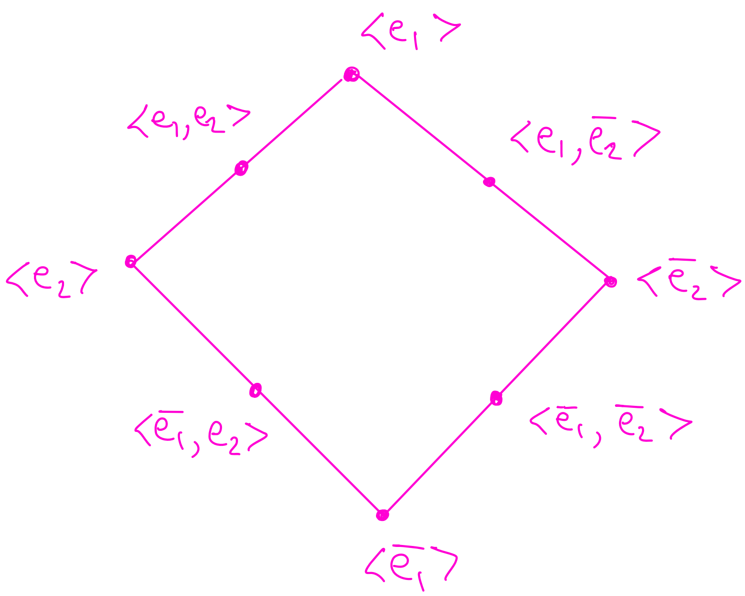
$\omega(e_i, \bar{e}_j) = \delta_{ij}$

$\omega(e_i, e_j) = 0 = \omega(\bar{e}_i, \bar{e}_j)$

Vertices $\longleftrightarrow$ Isotropic subspaces $0 \subsetneq W \subsetneq \mathbb{Q}^{2n}$ $(\omega|_W \equiv 0)$

Simplices $\longleftrightarrow$ Flags $0 \subsetneq W_0 \subsetneq W_1 \subsetneq \dots \subsetneq W_p \subsetneq \mathbb{Q}^{2n}$

Eg: $n = 2$ $\langle e_1, e_2, \bar{e}_1, \bar{e}_2 \rangle$



"apartment"

$[e_1, e_2, \bar{e}_1, \bar{e}_2]$

Thm [Solomon Tits] $\mathbb{T}^{\omega}_{2n} \simeq \vee S^{n-1}$

$St^{\omega}_{2n} := \tilde{H}_{n-1}\mathbb{T}^{\omega}_{2n}$ is generated by apartment classes

Sketch of Construction

Since $Sp_{2\cdot 1} = SL_2$, already have resolution for $St^{\omega}_{2\cdot 1}$.

$K(\mathbb{F}^{2n})$: simplicial complex

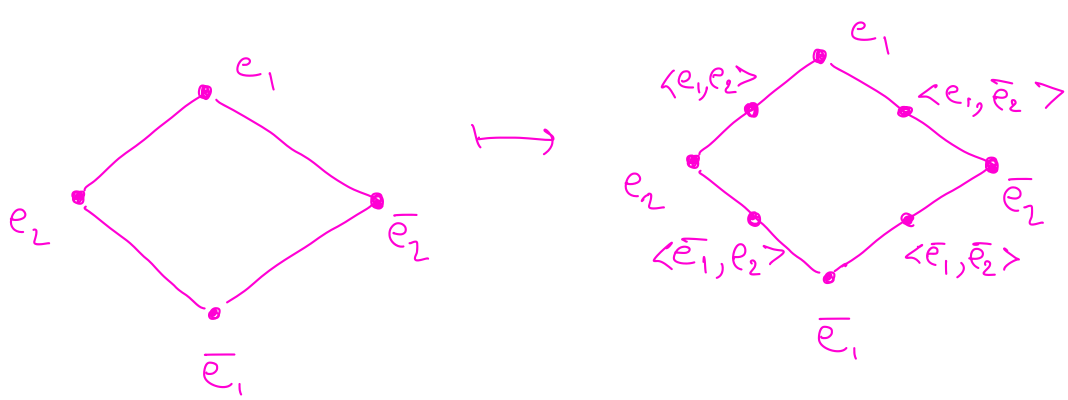
vertices $\leftrightarrow$ lines in $\mathbb{F}^{2n}$

simplices $\leftrightarrow$ all finite sets of vertices

$K(\mathbb{F}^{2n}) \simeq *$

$I^0(\mathbb{F}^{2n})$: simplices $\leftrightarrow$ isotropic sets of lines

Fact : $I^0(\mathbb{F}^{2n}) \simeq T^{\omega}_{2n}\mathbb{F}$



How do we get from I^0 to K?

$\hookrightarrow$ Genus filtration

$$I^0 \subset I^1 \subset \dots \subset I^n = K$$

I^g : lines that span a genus $\leq g$ subspace

$\rightsquigarrow$ spectral Sequence for a filtration ...

need to analyse $H_*(I^g / I^{g-1})$

Prop^n : $H_*(I^g / I^{g-1}) \cong \bigoplus_{\substack{W \subset \mathbb{F}^{2n} \\ \text{genus } W = g}} St(W) \otimes \underbrace{St^\omega(W^\perp)}_{\text{sympl. Steinberg of lower genus}}$

$\hookrightarrow * = n+g-1$

On Pg E^1 of spectral sequence:

$$0 \to H_{2n-1}(I^n/I^{n-1}) \to \dots \to H_{n+1}(I^2/I^1) \to H_n(I^1/I^0) \to H_{n-1}(I^0) \to 0$$

$K(\mathbb{F}^{2n}) \simeq * \implies$ exactness

$\rightsquigarrow$ Inductively use resolutions of each $H_*(I^g / I^{g-1})$ to get resolution of $St^\omega_{2n}\mathbb{F}$.

Resulting presentation of $St^\omega_{2n}\mathbb{F}$:

$$V_1 \to V_0 \to St^\omega_{2n}\mathbb{F} \to 0$$

$$V_0 = \bigoplus [v_1, w_1] \otimes \dots \otimes [v_n, w_n]$$

mutually perpendicular genus 1 subspaces

$$V_1 = \bigoplus [v_1, w_1] \otimes \dots \otimes [v_i, w_i, x_i] \otimes \dots \otimes [v_n, w_n]$$
$$+ \bigoplus [v_1, w_1] \otimes \dots \otimes [\underbrace{v_i, w_i, x_i, y_i}_{\text{span a genus 2 subspace}}] \otimes \dots \otimes [v_{n-1}, w_{n-1}]$$

Eg of Differential : $\partial [e_1, e_1+e_2, \bar{e}_1, \bar{e}_2]$

$$= [e_1, \bar{e}_1][e_1+e_2, \bar{e}_2] - [e_1+e_2, \bar{e}_1][e_2, \bar{e}_1-\bar{e}_2] + [e_1+e_2, \bar{e}_2][\bar{e}_1-\bar{e}_2, e_1]$$